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TWO-DIMENSIONAL CRACKS AT AN ANGLE TO AN INTERFACE

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Abstract—In this paper, we provide additional numerical results for the energy release rate and the stress intensity factors at the tip of a straight two-dimensional crack at an angle to an interface of a bimaterial system (He and Hutchinson, 1989, *Int. J. Solids Structures* **25**, 1053–1067). Three cases are considered: a semi-infinite crack under a pair of point loads, a finite crack under a pair of point loads, and a finite crack under uniform pressure. The formulations for each of these cases lead to a system of singular integral equations (He and Hutchinson, 1989, *Int. J. Solids Structures* **25**, 1053–1067) which can be solved numerically. The effects of the orientation of the crack, the distance between the crack and the interface, and the material properties, α, β, are investigated. The results are compared to the corresponding asymptotic results of Hutchinson *et al.* (1987) for the subinterface crack (crack parallel to the interface).

1. INTRODUCTION

There have been a number of investigations of the two-dimensional stress distribution about a crack on or near an interface [see e.g. the references in Hutchinson *et al.* (1987), He and Hutchinson (1989), Thouless *et al.* (1987), and Hutchinson and Suo (1992)]. Much of this work has focused on the need to better understand the structural integrity of layered systems and of materials containing interfaces, e.g. fiber-reinforced materials. An important part of these studies has been the investigation of the path taken by a crack as it approaches an interface. One of the first experimental and analytical investigations of the behavior of a crack near and parallel to an interface was by Thouless *et al.* (1987). This was followed by a study to investigate if a loaded crack close to and parallel to an interface (sub-interface crack) (Hutchinson *et al.*, 1987) will remain parallel to the interface.

In addition to these studies, experimental studies using indentation mechanics which has been used extensively to study the bulk properties of materials, in particular, ceramics (Lawn and Wilshaw, 1975; Ostojic and McPherson, 1987; Cook and Pharr, 1990), have been carried out to investigate the behaviour of cracks approaching interfaces (Evans *et al.*, 1986; Lardner *et al.*, 1990). Studies have also been carried out using electrical analogies to predict crack paths in composite materials in anti-plane stress (Madhusudhan *et al.*, 1990).

The basic investigation of a crack passing through or into an interface or approaching an interface at an angle was carried out by He and Hutchinson (1989). The special case of a crack approaching an interface at an angle was later studied by Swenson and Cao (1990) using a finite element analysis. An experimental study to determine the trajectories of cracks created by indentations near interfaces together with a finite element study to model the behavior of the indent crack was also carried out by Lardner *et al.* (1990). More recent investigations by Bhattacharya *et al.* (1992) have demonstrated that micro-indentation techniques can be used to evaluate the fracture energy of a bimaterial interface.

The purpose of this brief paper is to present a series of results for the case of a two-dimensional crack at an angle to an interface using the approach of He and Hutchinson (1989). We supplement their results by investigating the effect of the material parameters for the case of a semi-infinite crack under point loads and we show that the asymptotic sub-interface solution (Hutchinson *et al.*, 1987) is a surprisingly good approximation for a crack at a small angle to the interface. We also present results for a finite crack under a pair of point loads as a potential model of cracks formed under indents [for example, see Bhattacharya *et al.* (1992)] and, finally for comparison, we show the results for a finite

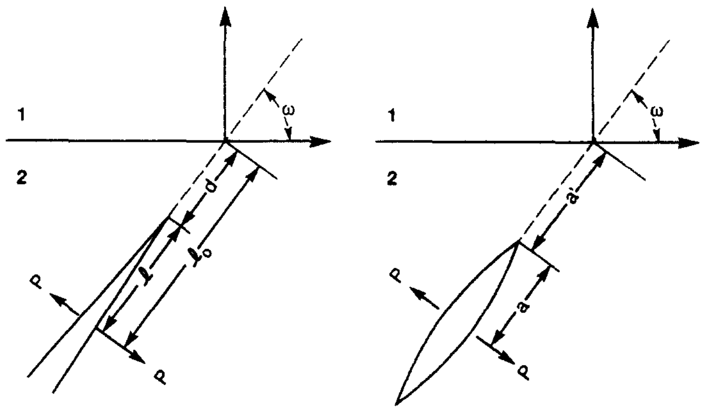

Fig. 1. (a) A semi-infinite crack under a pair of point loads at an angle to an interface (He and Hutchinson, 1989). (b) A finite crack under a pair of point loads at an angle to an interface. The cracks are in material 2.

crack with uniform pressure. We observe that for the case of a crack under a pair of point loads, the energy release rate as a function of the angle to the interface reaches a maximum value before approaching the asymptotic sub-interface solution for small angles. A similar result has been found for a crack parallel to an interface (Lu and Lardner, 1992).

2. SEMI-INFINITE CRACK AT AN ANGLE TO AN INTERFACE

The two-dimensional problem of a semi-infinite crack under a pair of point loads P and at an angle to an interface was first investigated by He and Hutchinson (1989) (Case C of their paper). Our notation and approach will follow that of He and Hutchinson (1989); our objective is to investigate further the effects of the material constants α, β and the angle ω on the energy release rate and mode mixity ($K_{\text{II}}/K_{\text{I}}$).

A two-dimensional semi-infinite crack at an angle ω to an interface between two different materials under a pair of point loads P is shown in Fig. 1(a). The distance of the crack tip from the interface is given by d, the distance of the loads from the interface by l_0 and from the crack tip by l. The crack is modeled by a dislocation density along the crack line from which the stresses on the crack surfaces induced by the density are constructed to obtain a singular integral equation for the density (He and Hutchinson, 1989). We follow He and Hutchinson (1989) and briefly comment on some of the steps in their derivation.

The stress components σ_θ and $\tau_{r\theta}$ at a point $z = t_2\,\mathrm{e}^{\mathrm{i}(\pi+\omega)}$ on the radial line $\theta = \pi+\omega$ induced by a dislocation at $s = \eta_2\,\mathrm{e}^{\mathrm{i}(\pi+\omega)}$ can be constructed and the final form is (He and Hutchinson, 1989),

$$\sigma_\theta(t_2)+\mathrm{i}\tau_{r\theta}(t_2) = 2\bar{B}(\eta_2)\,\mathrm{e}^{\mathrm{i}\omega}/(\eta_2-t_2)+B(\eta_2)G_1(\eta_2,t_2)+\bar{B}(\eta_2)G_2(\eta_2,t_2), \tag{1}$$

where

$$G_1(\eta_2,t_2) = \Pi\left[\frac{1}{z-\bar{s}}+\frac{\bar{s}-s}{(\bar{z}-s)^2}+\mathrm{e}^{2\mathrm{i}\omega}\frac{(\bar{s}-\bar{z})}{(\bar{z}-\bar{s})^2}\right],$$

$$G_2(\eta_2,t_2) = \Pi\left[\frac{1}{\bar{z}-s}+\frac{s-\bar{s}}{(z-\bar{s})^2}+\mathrm{e}^{2\mathrm{i}\omega}\frac{(s-\bar{s})(z+\bar{s}-2\bar{z})}{(z-\bar{s})^3}\right]+\Lambda\,\mathrm{e}^{2\mathrm{i}\omega}\frac{1}{z-\bar{s}},$$

where Λ is $(\alpha+\beta)/(1-\beta)$ and Π is $(\alpha-\beta)/(1+\beta)$; and α and β are Dundurs' parameters (Dundurs, 1969) which are defined as follows:

$$\alpha = \frac{\mu_1(k_2+1)-\mu_2(k_1+1)}{\mu_1(k_2+1)+\mu_2(k_1+1)},$$

$$\beta = \frac{\mu_1(k_2-1)-\mu_2(k_1-1)}{\mu_1(k_2+1)+\mu_2(k_1+1)}.$$

The applied tractions on the crack surface are combined with eqn (1) to obtain a singular integral equation for the determination of the dislocation density in the form,

$$\int_d^{\infty} [\{2\bar{B}(\eta_2)\,e^{i\omega}/(\eta_2-t_2)\}+B(\eta_2)G_1(\eta_2,t_2)+\bar{B}(\eta_2)G_2(\eta_2,t_2)]\,e^{i(\pi+\omega)}\,d\eta_2 = P(t_2), \tag{2}$$

where $P(t_2)$ represents the equal and opposite applied traction at location t_2 on the crack surfaces and $B(\eta_2)$ represents the value of the dislocation density at the location η_2. When the crack is under a pair of point loads, the term $P(t_2)$ is set equal to zero and the effect of the point load is included in the function $B(\eta_2)$ to reflect the singular behavior of the density.

The density, $B(\eta_2)$, as η_2 approach the crack tip behaves in the same manner as for a crack in a homogeneous material, i.e.

$$\lim_{\eta_2 \to d} B(\eta_2) \propto A_1 r^{-1/2}. \tag{3}$$

Far away from the crack tip and loads, the asymptotic limit for the solution of a crack touching the interface (He and Hutchinson, 1989) is used so that the singularity of $B(\eta_2)$ will behave in the asymptotic limit like that of a crack terminating at the interface

$$\lim_{\eta_2 \to \infty} B(\eta_2) \propto A_1^* r^{-\lambda^*}. \tag{4}$$

The exponent, λ^*, characterizes the asymptotic outer solution to the semi-infinite crack touching the interface for a loading on the crack that is confined to the vicinity of the tip and it is a function of ω and material constants α and β (Bogy, 1971; Hein and Erdogan, 1971; Fenner, 1976; He and Hutchinson, 1989, figure 5). For example, for a homogeneous material (no interface), the value of $\lambda^* = 3/2$.

The singular behavior of $B(\eta_2)$ near $\eta_2 = l_0$ where the point loads are acting must be consistent with the behavior,

$$\lim_{\eta_2 \to l_0} B(\eta_2) \propto \frac{-P\,e^{-i\omega}}{[2\pi^2(\eta_2-l_0)]}. \tag{5}$$

The behavior in eqn (5) can be obtained either from the solution for a point load acting on a free surface or from the corresponding crack problem for a homogeneous material.

To solve the governing integral equation for the dislocation density, eqn (2), the dislocation density $B(\eta_2)$ is written in a series of N terms of Chebyshev polynomials of the first kind of degree k together with terms including the singular behavior. Once the numerical solution for the dislocation density is obtained, the stress intensity factor at the end of the crack nearest to the interface can be obtained as follows:

$$K = K_I + iK_{II} = (2\pi)^{3/2}\,e^{i\omega} \lim_{t_2 \to d} (t_2-d)^{1/2}\bar{B}(t_2)$$

$$= \frac{2^{\lambda^*-1}(1+\xi_0)^{3/2-\lambda^*}P}{\sqrt{\pi l}} + (2\pi)^{3/2}2^{\lambda^*-1/2}\,e^{i\omega}\sum_{k=0}^{N-1}\tilde{a}_k, \tag{6}$$

where

$$\frac{l}{l_0} = \frac{1-\xi_0}{1+\xi_0}.$$

For the homogeneous case, $\lambda^* = 3/2$, and the stress intensity factor is

$$K = \frac{2}{\pi l} P, \tag{7}$$

where the $\bar{a}_k$ in eqn (6) are identically zero. The energy release rate is then calculated from the stress intensity factors,

$$G = \left[\frac{1-\nu_2}{2\mu_2}\right](K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2). \tag{8}$$

We consider first the three cases treated by He and Hutchinson (1989) with ω equal to 30°, 45° and 60° for two different combinations of materials; $\alpha = \pm 0.5$, $\beta = 0$. Figure 2 gives the non-dimensional energy release rate for $\omega = 60°$ showing results for $N = 2$ and 30 and the corresponding results from He and Hutchinson (1989), Fig. 9. The results for $N = 2$ are in close agreement with the results for $N = 30$ for both values of α. The results for $\alpha = 0.5$ are in close agreement with He and Hutchinson (1989). For $\alpha = -0.5$ and $l/l_0 > 0.5$, the energy release rate increases rapidly as l/l_0 approaches one and any small perturbation in the numerical scheme can give rise to differences in the results. In our model, we treat d, Fig. 1(a), as a variable dependent on l/l_0 whereas d was set equal to 0 in the He and Hutchinson (1989) formulation; we believe that this is the reason for the differences shown in Fig. 2 for the values in this range of l/l_0. Figure 3 shows the non-dimensional energy release rate versus l/l_0 for the three angles of approach. The calculations were carried out for $N = 40$ to verify the convergence of the results; we found that the results for $N = 30$ and $N = 40$ differ by 0.1%. We found close agreement for the ratio $K_{\mathrm{II}}/K_{\mathrm{I}}$ versus l/l_0 for the different values of ω with $\alpha = \pm 0.5$, $\beta = 0$ with the corresponding results of Fig. 7 in He and Hutchinson (1989).

Figure 4 illustrates the non-dimensional energy release rate for different orientations ω of the crack for the three cases of $l/l_0 = 0.29$, 0.5, 0.65. We observe in Fig. 4 that for $\alpha = 0.5$, the energy release rate approaches to the asymptotic sub-interface solution shown as a point on the $\omega = 0$ axis (Hutchinson *et al.*, 1987). The sub-interface solution is

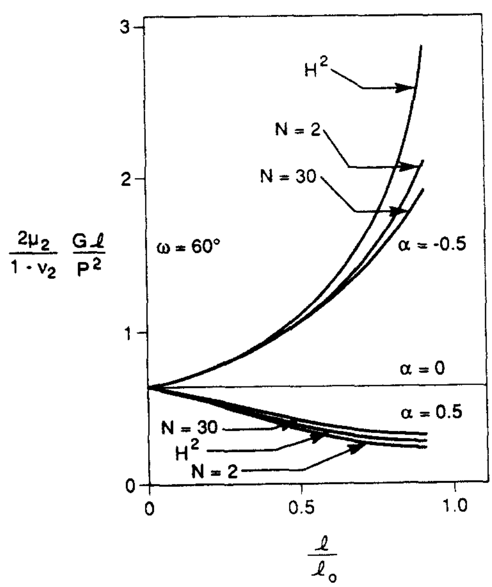

Fig. 2. The non-dimensional energy release rate for a straight semi-infinite crack at an angle of 60° to an interface for $\beta = 0$ and $N = 2$ and 30; the curve marked H^2 is from He and Hutchinson (1989), figure 9.

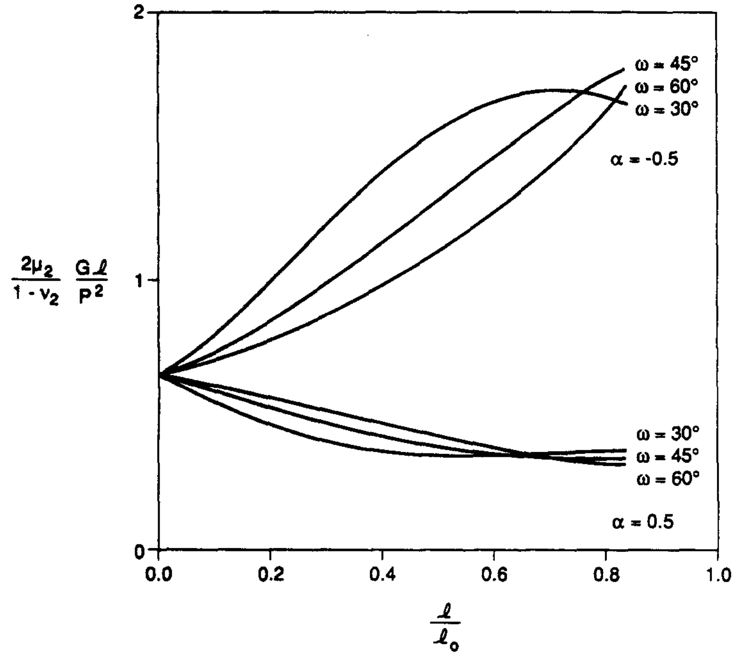

Fig. 3. Non-dimensional energy release rate for a straight semi-infinite crack with $\omega = 60°, 45°, 30°$ for $\beta = 0$.

independent of the distance of the crack tip from the interface when $\beta = 0$. Surprisingly, when $\alpha = 0.5$, the energy release rate remains fairly constant with changes in ω and is close to the sub-interface solution. The occurrence of the maximum in the energy release rate for $\alpha = -0.5$ in the curves of Fig. 5 is of interest. The maximum value of the energy release rate occurs when the crack tip is at an approximate distance of $0.18 l_0$ from the interface.

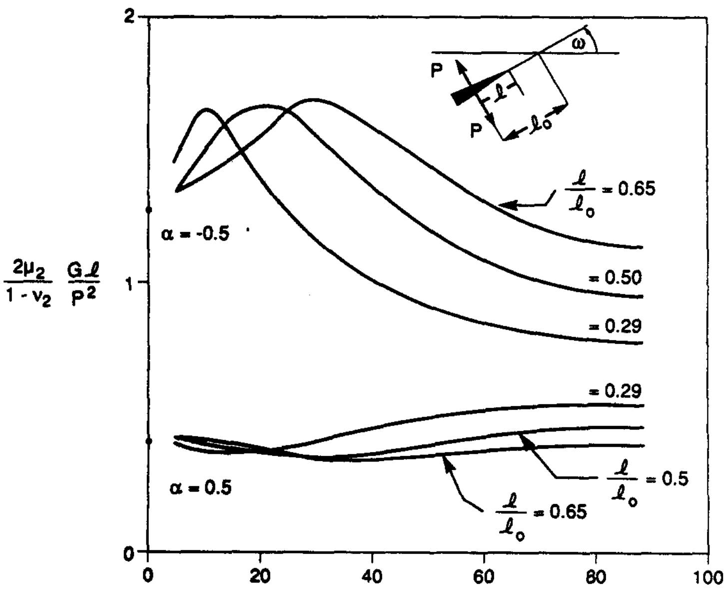

Fig. 4. Non-dimensional energy release rate for a semi-infinite crack at an angle ω to an interface for three values of l/l_0, $\beta = 0$; the point on the $\omega = 0$ axis is the sub-interface solution of Hutchinson *et al.* (1987) with $\beta = 0$.

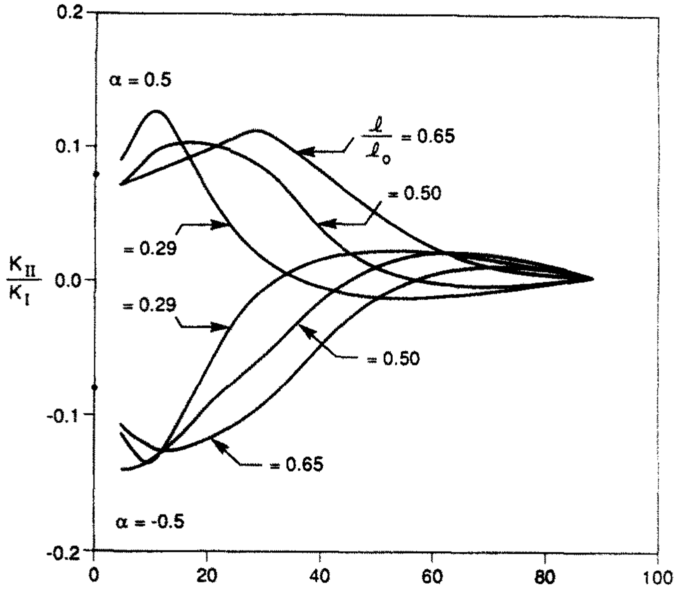

Fig. 5. Ratio of K_{II} to K_I for a semi-infinite crack at an angle ω to an interface for different l/l_0, $\beta = 0$; the point on the $\omega = 0$ axis is the sub-interface solution of Hutchinson *et al.* (1987) with $\beta = 0$.

Figure 5 illustrates the ratio of K_{II}/K_I for different orientation angles of the crack. In Fig. 5, we find that, for $\alpha = 0.5$, the ratio of K_{II}/K_I is slightly negative at large values of the angle ω, and then becomes positive. The ratio of K_{II}/K_I reaches a maximum value, then decreases and approaches the sub-interface solution. Similar results hold for $\alpha = -0.5$. Again we note that the sub-interface solution provides a good approximation for $\alpha = 0.5$ over a large range of values of ω. When the orientation angle ω of the crack is small, large values of N are needed to obtain convergence and $N = 40$ was used for $5° < \omega < 20°$.

In addition to the cases for $\alpha = \pm 0.5$, $\beta = 0$, we investigated for $\omega = 45°$ the cases for $\alpha = \pm 0.4$, $\beta = 0$, to study the effect of changing α with $\beta = 0$ and $\alpha = \pm 0.5$, $\beta = \pm 0.125$ to investigate the effect of β. We chose $\beta = \alpha/4$ in the spirit of the results found by Suga *et al.* (1988) that many material combinations follow this relationship. In general, the value of λ^* needed for the calculations is complex; however for $\omega = 45°$, λ^* is real.

Figure 6 illustrates the energy release rate for different values of α and β when $\omega = 45°$. We see in Fig. 6 that the effect of changing α and β is more significant for α negative.

3. FINITE CRACK AT AN ANGLE TO AN INTERFACE

In this section the results for a crack of finite length, $2a$ [Fig. 1(b)], under a pair of point loads or a uniform pressure at an angle to an interface are presented. The solution procedure follows that used for the semi-infinite case. We investigate the effect of fixing a', the distance of the nearest crack tip to the interface, and varying ω and the values of the material constants α, β on the solution.

Figures 7 and 8 show the normalized energy release rate, and the ratio of K_{II}/K_I for the cases $(a'/a) = 3, 1, 0.5$; N was >20 in the calculations. In Fig. 7, we find that the trends of the normalized energy release rate for a finite crack are similar to those of a semi-infinite crack for each of the three values of a'. Here again we find that the solution for the sub-interface crack is a good approximation for ω small, especially for $\alpha = 0.5$.

In Fig. 8, we find that for $\alpha = 0.5$, the ratio of K_{II}/K_I is slightly negative at large values of ω and then becomes positive. The ratio of K_{II}/K_I reaches a maximum value, then decreases

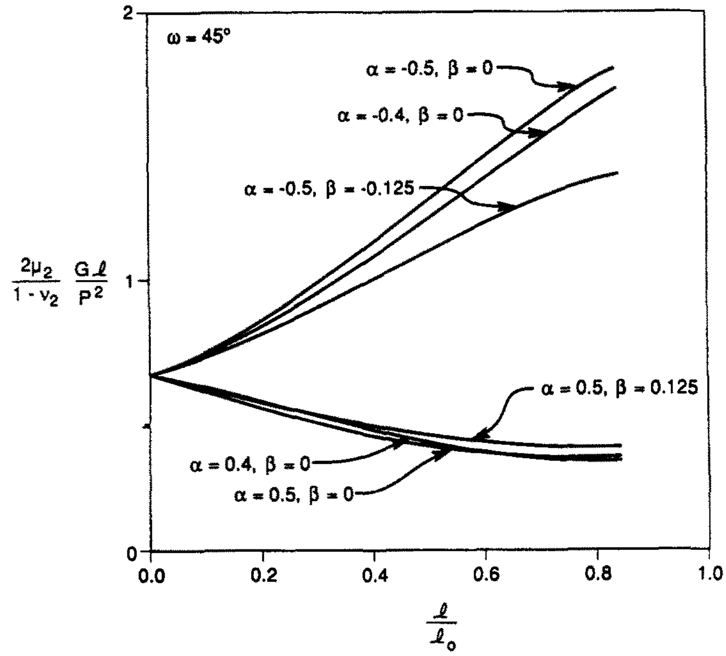

Fig. 6. Normalized energy release rate of a semi-infinite crack at an angle $\omega = 45°$ to an interface for different α, β.

and approaches the sub-interface solution. For $\alpha = -0.5$, the ratio of K_{II}/K_I is slightly positive at large values of ω and then becomes negative. We show the effect of α and β on the energy release rate when $\omega = 45°$ in Fig. 9. Here again we find the effect of β is not significant.

Finally, Figs 10 and 11 show the energy release rate and the ratio of K_{II}/K_I for the case of a crack under uniform pressure p as a function of ω. In Fig. 10, we observe that the energy release rate monotonically approaches the sub-interface solution as $\omega \to 0$ and no

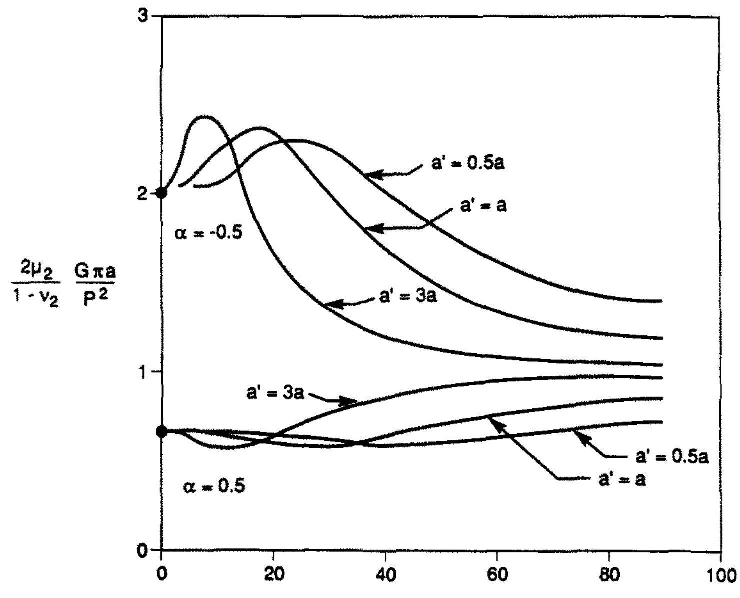

Fig. 7. Normalized energy release rate for a finite crack under a pair of point loads at an angle ω to an interface for three values of a' ($\beta = 0$); the point on the $\omega = 0$ axis is the sub-interface solution of Hutchinson *et al.* (1987) with $\beta = 0$.

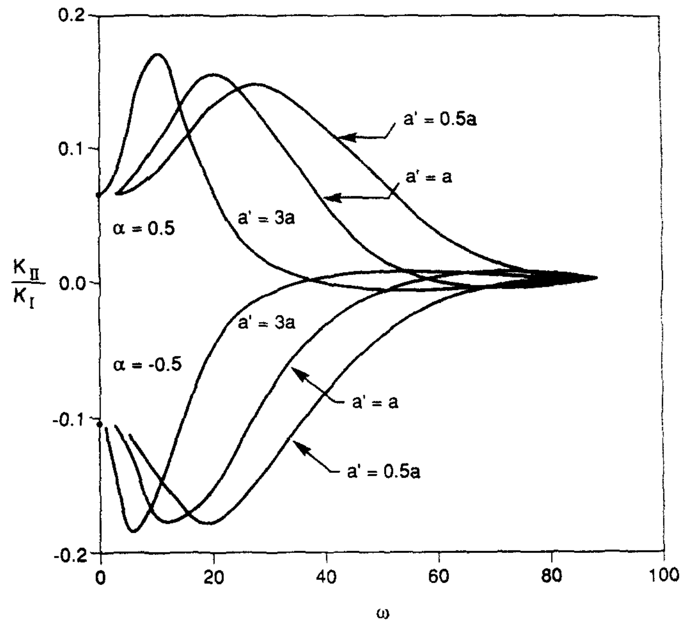

Fig. 8. The ratio of K_{II}/K_I of a finite crack at an angle ω to an interface for three values of a' ($\beta = 0$); the point on the $\omega = 0$ axis is the sub-interface solution of Hutchinson *et al.* (1987) with $\beta = 0$ ($\varepsilon = 0$).

maximum value occurs. In Fig. 11, we find that for $\alpha = 0.5$, the ratio of K_{II}/K_I is slightly negative for large angles of ω, while for $\alpha = -0.5$ the ratio of K_{II}/K_I is slightly positive for large angles of ω.

4. CONCLUSIONS

Three cases of a crack at an angle to an interface of a bimaterial system were considered: a semi-infinite crack under point loading near the crack tip, a finite crack under point

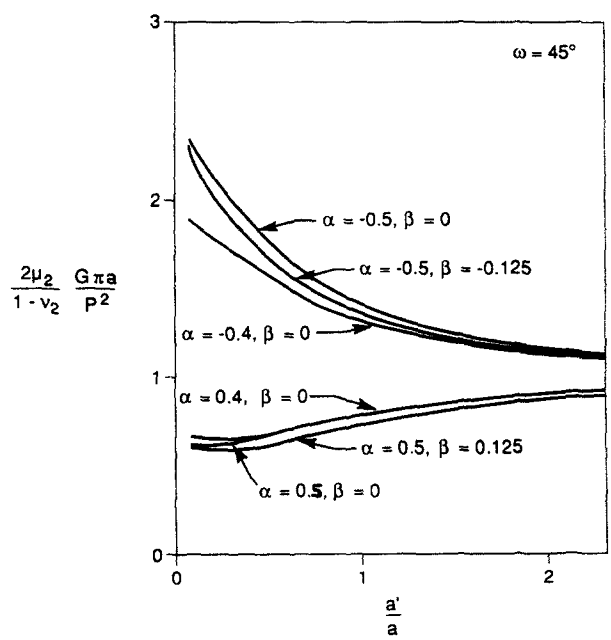

Fig. 9. Normalized energy release rate of a finite crack under a pair of point loads at an angle $\omega = 45°$ to an interface for different values of α, β with varying (a'/a).

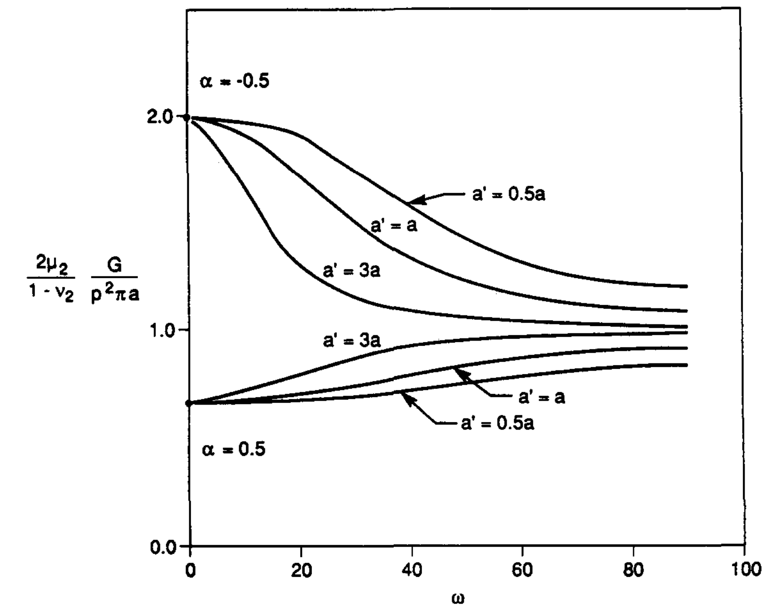

Fig. 10. Normalized energy release rate of a finite crack under uniform pressure p at an angle ω to an interface for three values of a' ($\beta = 0$).

loading, and a finite crack under uniform pressure. The formulation of the model following He and Hutchinson (1989) for these cases leads to a singular integral equation that requires a numerical scheme to solve for the unknown density function expressed in terms of Chebyshev polynomials. A collocation method is used to solve this system of singular integral equations (Erdogan and Gupta, 1972).

For a crack at an angle to an interface we found that when the crack tip is close to a stiffer material, $\alpha > 0$, the energy release rate will decrease as the interface is approached, and when the crack is close to a softer material, $\alpha < 0$, the energy release rate will increase as the interface is approached.

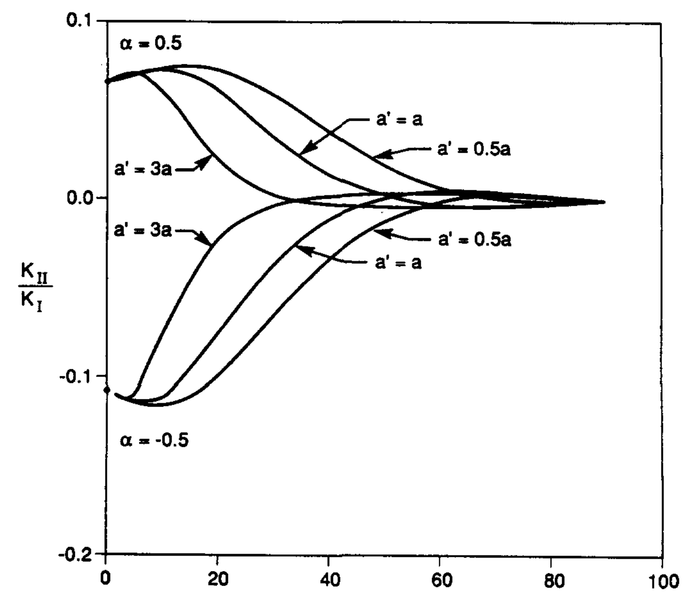

Fig. 11. The stress intensity factors of a finite crack under uniform pressure p at an angle ω to an interface for three values of a' ($\beta = 0$).

For a crack whose crack tip is at a fixed distance from the interface at an angle ω, we find that when the cracks are under concentrated loading, for $\alpha < 0$, the energy release rate increases to a maximum value as the orientation of the crack, ω, decreases. After the maximum value of energy release rate is reached, the value of the energy release rate decreases as ω decreases and approaches the sub-interface solution of Hutchinson *et al.* (1987). Similar results apply when $\alpha > 0$.

In the case of a finite crack under uniform pressure, we find that for $\alpha < 0$ the energy release rate increases monotonically and approaches to the sub-interface solution as ω decreases, with similar behavior when $\alpha > 0$. No maximum or minimum occurs in the results for the energy release rate for a finite crack under uniform pressure.

We find that in all three cases for a crack approaching an interface, for $\alpha < 0$, the ratio of K_{II}/K_{I} begins with a slightly positive value and then becomes negative as the interface is approached; for $\alpha > 0$, the ratio of K_{II}/K_{I} starts with a slightly negative value and then becomes positive. In the analysis of the results, we conclude that for $\alpha > 0$ the interface will tend to turn the trajectory of an advancing crack to a direction parallel to the interface, and for $\alpha < 0$ the trajectory of the crack will turn into the interface.

For a pair of point loads acting on a finite crack we find that the energy release rate for small values of ω can be higher than that of a sub-interface crack with $\omega = 0$. We also find in each of the three cases that the solution for the sub-interface crack (with $\beta = 0$) provides a good approximation to the solution when ω is small; for $\alpha > 0$, the sub-interface solution provides a fairly good approximation over a fairly large range of values of ω. The effect of $\beta \neq 0$ on the value of the energy release rate and (K_{II}/K_{I}) in general is small. Additional numerical results can be found in Chen (1991).

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